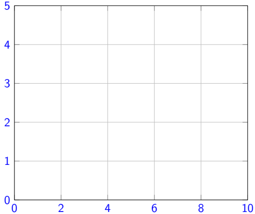
Chapter 4.2: The Mean Value Theorem

Rolle's Theorem

If $f(x)$ is a function so that

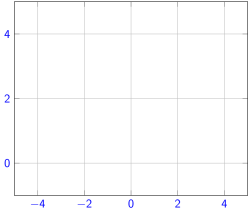
- $f(a) = f(b)$
- continuous for $a \leq x \leq b$
- differentiable for $a < x < b$

then for some c where $a < c < b$ we have $f'(c) = 0$.



Idea: f must achieve and absolute max/min. These are at critical points and at least one is not an endpoint so must be where derivative of f is 0.

Differentiable is necessary



No c with $f'(c) = 0$. $|x|$ for $x \in [-2, 2]$

Example: $f(x) = x^3 - 7x$, $a = -3$, $b = 1$
Notice $f(-3) = f(1) = -6$.
Noe $f'(x) = 3x^2 - 7$. We want c such that $0 = f'(c) = 3c^2 - 7$.
It gives $c = \pm\sqrt{\frac{7}{3}}$. Since $a < c < b$,
$c = -\sqrt{\frac{7}{3}}$.

Mean Value Theorem

If $f(x)$ is a function that is

- continuous for $a \leq x \leq b$
- differentiable for $a < x < b$

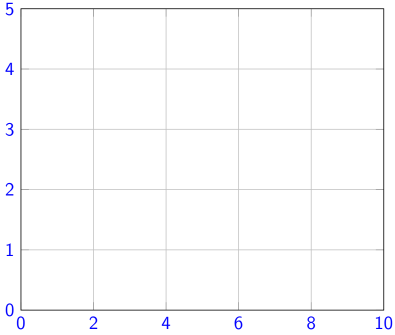
then for some c where $a < c < b$ we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Idea: Instantaneous rate of change at c is equal to the average rate of change for $a \leq x \leq b$.

Rolle's Theorem, but slightly tilted.

Example:



Note: The mean value theorem has been used by law enforcement to catch speeders!

Examples $f'(c) = \frac{f(b)-f(a)}{b-a}$ for $c \in (a, b)$

Verify the mean value theorem for $h(x) = \ln(x-1)$ for $a = 2$ and $b = e + 1$

The average rate of change is

$$\frac{h(e+1) - h(2)}{e+1-2} = \frac{1}{e-1}$$

and the derivative is $h'(x) = 1/(x-1)$. Consequently, we wish to solve

$$\frac{1}{e-1} = \frac{1}{c-1} \quad \longrightarrow \quad c = e$$

$f(x) = \frac{2x}{2x+1}$, $a = 0$, $b = 1$

The average rate of change is

$$\frac{f(b) - f(a)}{b-a} = \frac{\frac{2}{3} - 0}{1-0} = \frac{2}{3}$$

Now the derivative

$$\begin{aligned} f'(x) &= \frac{2(2x+1) - 2x(2)}{(2x+1)^2} \\ &= \frac{2}{(2x+1)^2} \end{aligned}$$

Solve $f'(c) = \frac{2}{3}$.

$$\frac{2}{3} = \frac{2}{(2c+1)^2} \qquad 3 = (2c+1)^2$$

$$c = \frac{\pm\sqrt{3} - 1}{2} \qquad c = \frac{\sqrt{3} - 1}{2}$$

Consequences of $f'(c) = \frac{f(b)-f(a)}{b-a}$ for $c \in (a, b)$

- If $f'(x) = 0$ for all $a < x < b$, then f is constant on (a, b). That is $f(x) = c$.
  Let $a < y < x < b$. By mean value theorem
  $$\frac{f(x) - f(y)}{x - y} = f'(z) = 0.$$
  This implies $f(x) = f(y)$.

- If $f'(x) = g'(x)$ for all $x \in (a, b)$, then there is a constant C such that
  $$f(x) = g(x) + C$$
  for all $x \in (a, b)$.
  Define $G = f - g$. Then for any $x \in (a, b)$, we have
  $$G'(x) = f'(x) - g'(x) = 0$$
  By the previous point, it follows that $f(x) - g(x) = G(x) = C$ for all x in (a, b).

If two derivatives are equal, they came from functions which differ by a constant.

Examples

Consider

$$f(x) = \ln x \qquad g(x) = \ln(ax)$$

Show $f'(x) = g'(x)$ and determine C so that $f(x) = g(x) + C$.

$$f'(x) = \frac{1}{x} \qquad g'(x) = \frac{1}{ax} \cdot a = \frac{1}{x}$$

Notice $f(x)$ and $g(x)$ differ by the same constant everywhere. So we can pick for example $x = 1$ and get

$$C = f(1) - g(1) = \ln 1 - \ln a = -\ln a$$

Hence $f(x) = g(x) - \ln a$.
We can check that
$g(x) = \ln(ax) = \ln a + \ln x$

Find all function $f(x)$ whose derivative is $\cos(x)$ on $(-\infty, \infty)$.
Let $g(x) = \sin(x)$. Then

$$g'(x) = \cos(x) = f'(x)$$

on $(-\infty, \infty)$ and so

$$f(x) = \sin(x) + C$$

for some constant C.

Examples

Find all function $f(x)$ whose derivative is $1/x$ on $(0, \infty)$ and $f(1) = 0$.

Let $g(x) = \ln(x)$, then

$$g'(x) = 1/x = f'(x)$$

on $(0, \infty)$ and thus $f(x) = \ln(x) + C$. Since $0 = f(1)$, it follows that the only function with this property is

$$f(x) = \ln(x).$$

Given that the velocity is $v = 32t - 2$ on $(0, 1)$ and $s(1/2) = 4$, find an equation for the position function $s(t)$ for t in $(0, 1)$.

Let $g(t) = 16t^2 - 2t$, then

$$g'(t) = 32t - 2 = v$$

on $(0, 1)$ and so

$$s(t) = 16t^2 - 2t + C.$$

Plugging in $4 = s(1/2)$ then yields

$$4 = s(1/2) = 4 - 1 + C = 3 + C$$

and so $C = 1$. Therefore, the position on $(0, 1)$ is

$$s(t) = 16t^2 - 2t + 1.$$